2022.M29C

2022L003A1EC

Coimisiún na Scrúduithe Stáit
State Examinations Commission

Leaving Certificate Examination

Mathematics

Paper 1

Higher Level

2 hours 30 minutes

220 marks

Examination Number	
Day and Month of Birth	For example, 3rd February is entered as 0302
Centre Stamp	

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Instructions

There are **two** sections in this examination paper.

Section A	Concepts and Skills	120 marks	6 questions
Section B	Contexts and Applications	100 marks	4 questions

Answer questions as follows:

- any **four** questions from Section A – Concepts and Skills
- any **two** questions from Section B – Contexts and Applications.

Write your Examination Number in the box on the front cover.

Write your answers in blue or black pen. You may use pencil in graphs and diagrams only.

This examination booklet will be scanned and your work will be presented to an examiner on screen. Anything that you write outside of the answer areas may not be seen by the examiner.

Write all answers into this booklet. There is space for extra work at the back of the booklet. If you need to use it, label any extra work clearly with the question number and part.

The superintendent will give you a copy of the *Formulae and Tables* booklet. You must return it at the end of the examination. You are not allowed to bring your own copy into the examination.

You will lose marks if your solutions do not include relevant supporting work.

You may lose marks if the appropriate units of measurement are not included, where relevant.

You may lose marks if your answers are not given in simplest form, where relevant.

Write the make and model of your calculator(s) here:

Section A	Concepts and Skills	120 marks

Answer **any four** questions from this section.

Question 1 **(30 marks)**

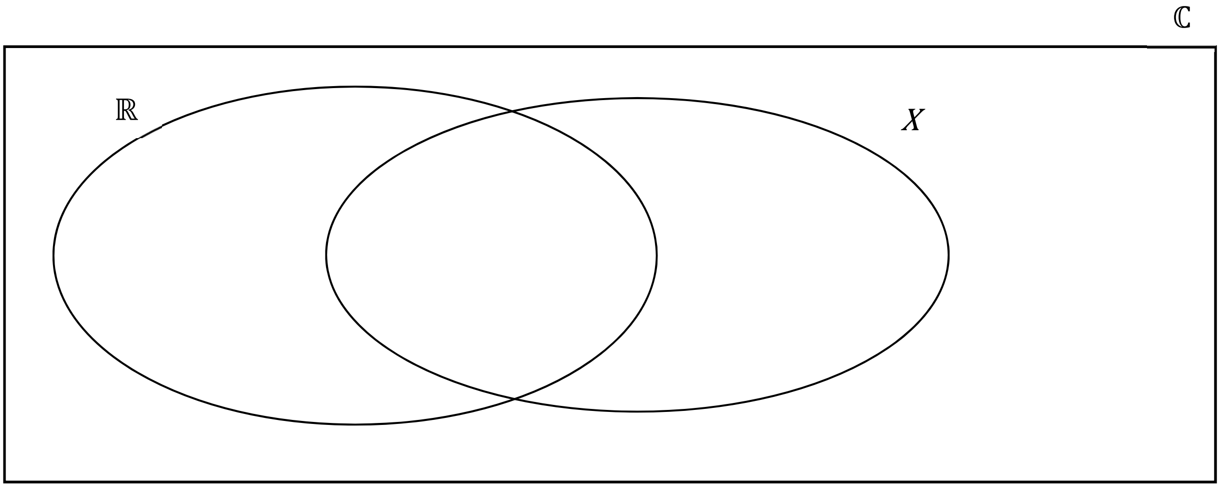
(a) Write the following complex number in the form $m + ni$, where $m, n \in \mathbb{R}$ and $i^2 = -1$:

$$(3 - 5i)(2 + 4i)$$

(b) Use De Moivre's theorem to write the following complex number in the form $p + qi$, where $p, q \in \mathbb{R}$ and $i^2 = -1$:

$$(3 - 3i)^6$$

(c) $\mathbb{R}$ is the set of real numbers. $\mathbb{C}$ is the set of complex numbers.
X is the set of complex numbers whose real and imaginary parts are both rational.
These sets are shown in the Venn diagram below.



Six numbers are shown in the table below, where $i^2 = -1$.

Write each number in the correct region on the Venn diagram above, to show which region each number lies in.

$\sqrt{17}$	$3i$	$3 + 2i$
$\pi + 5i$	$2{\cdot}5$	$\cos 180° + i \sin 180°$

Question 2 **(30 marks)**

(a) **(i)** Multiply out and simplify $(2m+1)^2$.

Any positive odd number can be written in the form $2k+1$, where $k \in \mathbb{Z}$ and $k \geq 0$, and any number in this form must be odd.

(ii) Using your answer to **part (a)(i)**, show that the square of any odd number is also an odd number.

(iii) Assume that t is a positive odd number.
Prove by induction that t^n is odd, for all $n \in \mathbb{N}$.

There is space for more work on the next page.

(b) Write the following expression as a single fraction in its simplest form, where $x \in \mathbb{R}$:

$$\frac{2}{3x-5} - \frac{6}{x}$$

Question 3 **(30 marks)**

(a) -5 and $\frac{3}{2}$ are the solutions of the following equation, where $x \in \mathbb{R}$ and $b, c \in \mathbb{Q}$:

$$x^2 + bx + c = 0$$

Find the value of b and the value of c.

$b =$ ______________ $c =$ ______________

(b) Divide $-6x^3 - 4x^2 + 44x + 56$ by $2x + 4$.

(c) Solve the following inequality for $x \in \mathbb{R},\ x \neq \frac{3}{2}$:

$$-4 \leq \frac{3x+5}{2x-3}$$

Question 4 **(30 marks)**

(a) A line l and a curve c have equations as follows, where $x, y \in \mathbb{R}$:

$$l: \quad x - 3y - 1 = 0$$
$$c: \quad x^2 + 4y^2 = 1$$

Find the co-ordinates of the points of intersection of l and c.

Points of intersection: (,) and (,)

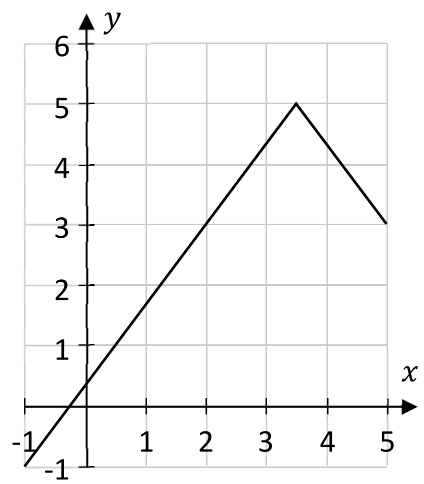
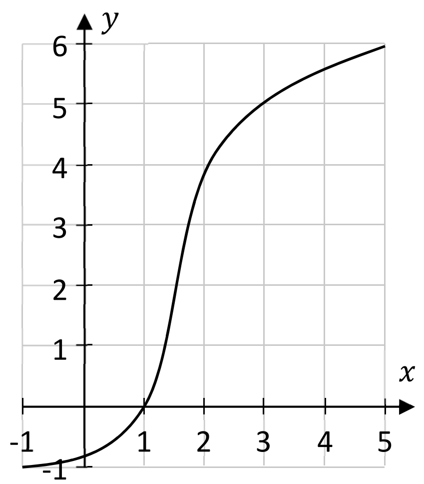
(b) The functions g and h are as follows, for $x \in \mathbb{R}$:

$$g(x) = x(2x-1)+3$$
$$h(x) = 5x+7$$

Write the composite function $h(g(x))$ in the form ax^2+bx+c, where $a, b, c \in \mathbb{Z}$.

(c) The graphs of two functions, $j(x)$ and $k(x)$, are shown below, for $-1 \leq x \leq 5$, $x \in \mathbb{R}$.

Use the graphs to estimate the value of $k(j(2))$.
Show your work on the graphs.

Graph of $j(x)$	**Graph of $k(x)$**
	

$k(j(2)) =$ ____________

Question 5 **(30 marks)**

The function $g(x)$ is defined as follows, for $x \in \mathbb{R}$:

$$g(x) = 2 - \frac{1}{e^x}$$

(a) Find the value of $g(4)$. Give your answer correct to 4 decimal places.

(b) Find the value of $\lim_{x \to \infty} g(x)$.

(c) Solve the following equation in x:

$$g(x) = \frac{1}{4}$$

Give your answer in the form $\ln a - \ln b$, where $a, b \in \mathbb{N}$.

(d) Find the value of the following integral:

$$\int_0^5 g(x)\ dx$$

Give your answer in the form $a + e^{-b}$, where $a, b \in \mathbb{N}$.

Question 6 **(30 marks)**

(a) The function f is defined as follows, for $x \in \mathbb{R}$:

$$f : x \mapsto x^3 - 6x^2 + 3$$

Use calculus to find the co-ordinates of each of the following points:

(i) the local minimum point of f

(ii) the local maximum point of f

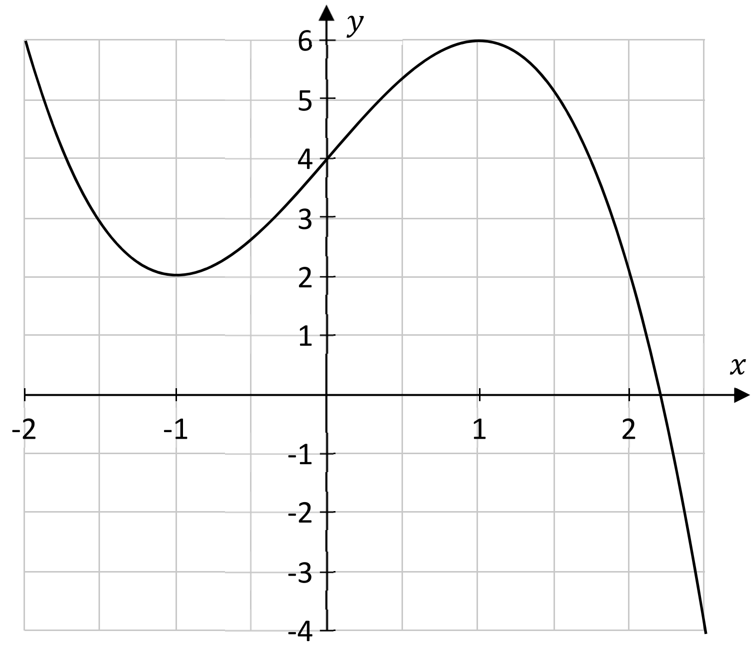
(iii) the point of inflection of f

(i) Local minimum: (,)

(ii) Local maximum: (,)

(iii) Point of inflection: (,)

(b) The diagram below shows the graph of the cubic function $g(x)$ for $-2 \leq x \leq 2{\cdot}5$, $x \in \mathbb{R}$.
There is a local minimum point at $(-1, 2)$ and a local maximum point at $(1, 6)$.



(i) Write down the range of values of x for which $g(x)$ is increasing.

(ii) Explain why the function $g(x)$ does **not** have an inverse, for $-2 \leq x \leq 2{\cdot}5$, $x \in \mathbb{R}$.

Section B   Contexts and Applications   100 marks

Answer **any two** questions from this section.

Question 7 **(50 marks)**

(a) The sale and resale of a particular car can be described by the following model.

The car is sold at the beginning of each year.
The initial price of the car, at the beginning of Year 0, is €$24\,000$.
Each time it is resold, the price is 75% of the previous year's price.

(i) Complete the table below to show the price of the car in the first 5 years of the model.

Year	**0**	**1**	**2**	**3**	**4**
Price of car at beginning of year (€)	$24\,000$		$13\,500$		

(ii) Ignoring rounding, write the price of the car at the beginning of Year 10 of the model in the form:

$$a \times (b^n)$$

where $a, b, n \in \mathbb{R}$.

(iii) Year p is the first year of the model for which the price of the car at the beginning of the year is less than €1000, where $p \in \mathbb{N}$. Find the value of p.

(iv) Use the formula for the sum of a geometric series to find the total amount of money paid for the car over the first 20 years of the model. Do **not** include the price for Year 20 in your solution. Give your answer correct to the nearest ten euro.

(b) The price of a different car, x years after it was initially sold, is given by:

$$P(x) = 25\ 000 \times (0{\cdot}8^x)$$

where $P(x)$ is the price in euro, and $x \in \mathbb{R}$.

Use integration to work out the average price of this car over the first 10 years after it was initially sold. Give your answer correct to the nearest euro.

This question continues on the next page.

(c) Explain what is meant by the present value, P, of a payment of F euro to be received in t years' time, where the annual rate of interest is i (and $P, F, t, i \in \mathbb{R}$).

(d) Mark and Aisha buy a house for €$280\ 000$.

(i) They pay 15% of the price of the house from their savings.
They take out a mortgage to pay for the remainder of the price.
Their mortgage is for 25 **years**, at an interest rate of $0{\cdot}36\%$ per **month**.

Work out their monthly repayments, given that they will make an equal repayment at the end of each month. Give your answer in euro, correct to the nearest cent.

(ii) The value of the house is expected to increase at a rate of $r\%$ each year, where $r \in \mathbb{R}$. Ten years after Mark and Aisha buy the house for €$280\ 000$, it is estimated that it will have a value of €$350\ 000$.

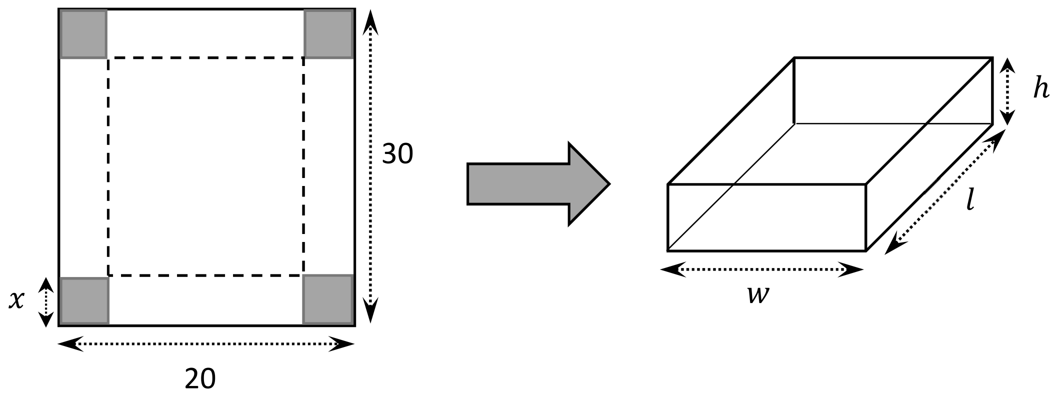
Use this to work out the value of r.
Give your answer as a percentage, correct to 2 decimal places.

Question 8 **(50 marks)**

A factory makes open rectangular boxes (with no lid) of various lengths, widths, and heights.

Each box is made from a rectangular sheet of cardboard.
A square of side x cm, where $x \in \mathbb{R}$, is removed from each corner of the cardboard.
The cardboard is then folded along the dotted lines, as shown in the diagram below.
Boxes of different dimensions (l, w, and h) can be made by using sheets of cardboard of different dimensions, or by using different values of x.

The diagram (not to scale) shows a sheet of cardboard with dimensions 20 cm × 30 cm, and the resulting box.



(a) **(i)** Fill in the boxes below to show the height (h), width (w), and length (l) of this box, in terms of x. The width is already done.

$h =$ ☐ cm $\quad w =$ $20 - 2x$ cm $\quad l =$ ☐ cm

(ii) Write the volume, $V(x)$, of this box in the form:

$$V(x) = ax^3 + bx^2 + cx$$

where $a, b, c \in \mathbb{Z}$, $0 < x < 10$, and $V(x)$ is in cm^3.

(b) A second box is being made from a rectangular sheet of cardboard with different dimensions. The volume of this box, $W(x)$, is given by:

$$W(x) = 4x^3 - 64x^2 + 240x$$

where $0 < x < 6$ and $W(x)$ is in cm^3.

Use calculus to find the value of x that gives a maximum value of $W(x)$.
Give your answer correct to 1 decimal place.

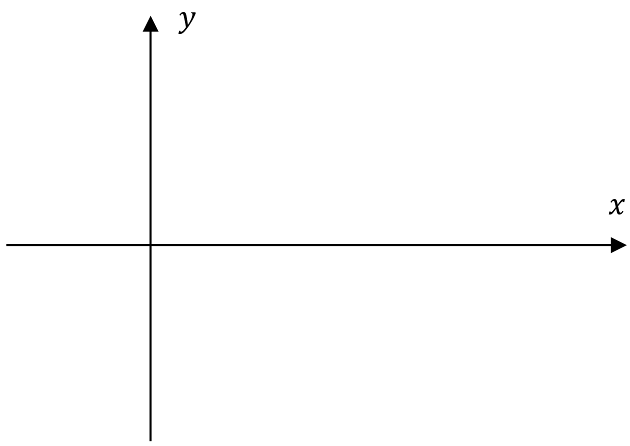
This question continues on the next page.

(c) A third box is made from a rectangular sheet of different dimensions. The volume, in cm^3, of this box is given by $U(x)$, for an appropriate range of values of $x \in \mathbb{R}$, where:

$$U(x) = x(42 - 2x)(30 - 2x)$$

(i) Write down the three roots of $U(x) = 0$.

(ii) Sketch the graph of $y = U(x)$ on the axes below.
On your sketch, show all three roots of $U(x) = 0$ **and** mark in the value of each root.



(iii) The function $U(x)$ gives the volume of this third box for a particular range of values of $x \in \mathbb{R}$. Write down this range of values of x.

(d) A fourth box is made from a square sheet of cardboard.

The volume, in cm^3, of this box is given by $T(x)$, where $0 < x < \frac{A}{2}$ and $A \in \mathbb{N}$:

$$T(x) = x(A - 2x)(A - 2x)$$

Use calculus to show that the maximum value of $T(x)$ is at $x = \frac{A}{6}$.

Question 9 **(50 marks)**

(a) As part of his work, Fred monitors the number of visitors to various websites.
One particular website has:

- 2506 unique visitors on one week (Week 0), and
- 2734 unique visitors the following week (Week 1).

It is possible to model $V(x)$, the number of unique visitors to this website in Week x, using different types of function, for $x \in \mathbb{Z}$, $x > 0$. Each of these models uses the fact that $V(0) = 2506$, and that there were 2734 unique visitors in Week 1. These models will usually give different values of $V(x)$ for values of x other than 0 or 1.

(i) Use a **linear** function to model the number of unique visitors per week to this website. That is, write $V(x)$ in the form $V(x) = ax + b$, where $a, b \in \mathbb{R}$.

(ii) Use a **quadratic** function to model the number of unique visitors per week to this website. That is, write $V(x)$ in the form $V(x) = cx^2 + dx + f$, where $c, d, f \in \mathbb{R}$.
(Note that there are an infinite number of different possible quadratic functions.)

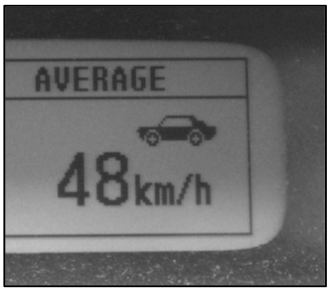
(iii) Use an **exponential** function to model the number of unique visitors per week to this website. That is, write $V(x)$ in the form $V(x) = k \times e^{px}$, where $k, p \in \mathbb{R}$.
Give the value of p correct to 3 decimal places.

(b) Fred drives to work. It is 12 km from Fred's home to work.
His average speed on the way to work is 40 km/h.
Because of traffic, his average speed on the way home is less than this.
His overall average speed for the two journeys (to work and home again) is 25 km/h.

Work out Fred's average speed on the way home from work.
Give your answer correct to 1 decimal place.

This question continues on the next page.

(c) Fred's car has a display that shows the average speed for the journey so far, in km/h. At the start of each new journey, the average speed is reset to 0. It changes as the journey continues.



The average speed is given in km/h, **correct to the nearest whole number**.

(i) What is Fred's average speed at the instant when the display changes from 47 km/h to 48 km/h?

Fred starts a long journey. After a brief time travelling at varying speeds, he joins a motorway and begins travelling at a constant speed of 120 km/h.

Some time later, he notices that the display showing his average speed for the journey changes from 115 km/h to 116 km/h.

At this instant, Fred has been travelling for a total of T hours and has travelled a total distance of D kilometres.

(ii) Write D in terms of T.

(iii) Exactly 20 minutes later, the display changes from 116 km/h to 117 km/h. Fred is still travelling at 120 km/h.

Use this information to write another equation linking D and T.

(iv) Fred continues to drive on the motorway at a constant speed of 120 km/h.
At the instant that the display changes from 116 km/h to 117 km/h, Fred has travelled a total of 174·75 km in 1·5 hours.

x minutes later, the display changes to 118 km/h, as shown in the following equation, where $x \in \mathbb{R}$:

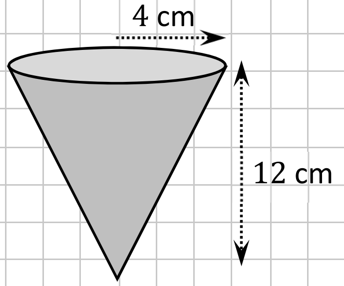
$$1{\cdot}5 + \frac{x}{60} = \frac{174{\cdot}75 + 120\left(\frac{x}{60}\right)}{117{\cdot}5}$$

Solve this equation to find the value of x, **and** hence find the total distance travelled by Fred when the display changes to 118 km/h.

Question 10 **(50 marks)**

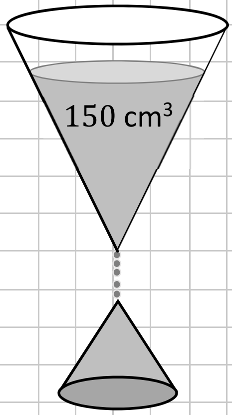
A funnel in the shape of an inverted right circular cone has a height of 12 cm and a radius of 4 cm.

(a) The funnel is full of sand. Find the volume of sand in the funnel, correct to the nearest cm^3.



(b) The sand drains from the funnel as shown below.
As the sand falls, it forms a right circular cone on a level surface under the funnel.
The height of the cone of fallen sand is equal to its diameter.

Find the height of the cone of fallen sand, when there are 150 cm^3 of sand left in the funnel.
Give your answer correct to 2 decimal places.

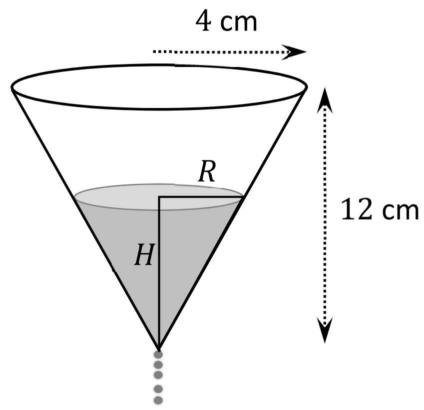


(c) The curved surface area of the cone of fallen sand, A, is given by $A = \pi r^2\sqrt{5}$ cm^2,
where $r \in \mathbb{R}$ is the radius of the cone of fallen sand, in cm.

Find the rate at which this curved surface area is increasing with respect to r,
when $r = 3$ cm.

(d) After a certain time, the remaining sand in the funnel is in the shape of a cone with a height of H cm and a radius of R cm.

Write H in terms of R.



(e) The sand drains from the funnel at a steady rate of 3 cm^3 per second.

Find the rate at which the radius of the remaining sand in the funnel is decreasing when there are 8π cm^3 of sand left in the funnel. Give your answer in cm^3 per second.

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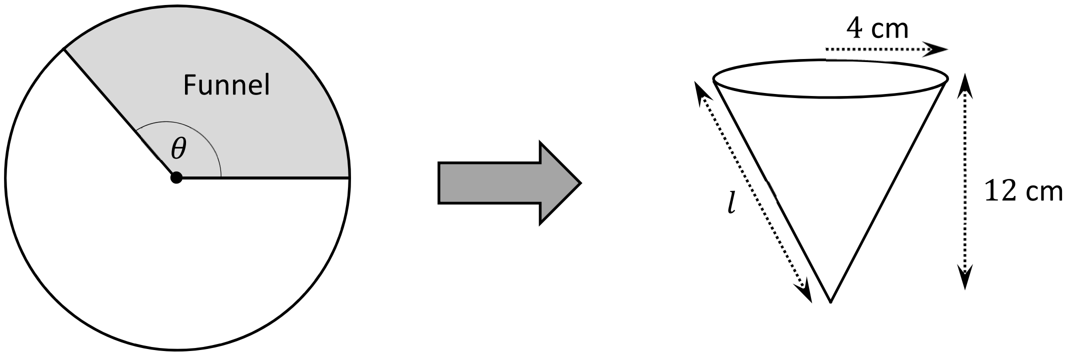
(f) The curved surface of the funnel is cut from a circular piece of metal of radius l.



(i) Find l, the slant height of the funnel. Give your answer in cm, in surd form.

(ii) Using your value for l, work out the size of the angle marked θ in the diagram above. Give your answer correct to the nearest degree.

Page for extra work.
Label any extra work clearly with the question number and part.

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Leaving Certificate – Higher Level

Mathematics Paper 1

2 hours 30 minutes