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Coimisiún na Scrúduithe Stáit
State Examinations Commission

Leaving Certificate Examination

Mathematics

Paper 1

Higher Level

2 hours 30 minutes

300 marks

Examination Number	
Day and Month of Birth	For example, 3rd February is entered as 0302
Centre Stamp	

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Instructions

There are **two** sections in this examination paper.

Section A	Concepts and Skills	150 marks	6 questions
Section B	Contexts and Applications	150 marks	4 questions

Answer questions as follows:

- any **five** questions from Section A – Concepts and Skills
- any **three** questions from Section B – Contexts and Applications.

Write your Examination Number in the box on the front cover.

Write your answers in blue or black pen. You may use pencil in graphs and diagrams only.

This examination booklet will be scanned and your work will be presented to an examiner on screen. Anything that you write outside of the answer areas may not be seen by the examiner.

Write all answers into this booklet. There is space for extra work at the back of the booklet. If you need to use it, label any extra work clearly with the question number and part.

The superintendent will give you a copy of the *Formulae and Tables* booklet. You must return it at the end of the examination. You are not allowed to bring your own copy into the examination.

In general, diagrams are not to scale.

You will lose marks if your solutions do not include relevant supporting work.

You may lose marks if the appropriate units of measurement are not included, where relevant.

You may lose marks if your answers are not given in simplest form, where relevant.

Write the make and model of your calculator(s) here:

Section A	Concepts and Skills	150 marks

Answer **any five** questions from this section.

Question 1 **(30 marks)**

(a) Solve the simultaneous equations

$$\begin{aligned} x + y &= 2 \\ y + 2z &= 12 \\ 5x + 7y + z &= 13 \end{aligned}$$

(b) The terms of the binomial expansion of

$$\left(x + \frac{k}{x}\right)^{10}$$

are written in **ascending** powers of x, where $x, k \in \mathbb{R}$, and k is a constant.

(i) Find, in terms of k and x, the third term in this expansion.

(ii) The coefficient of x^4 in this expansion is 7680. Find the value of k.

Question 2 **(30 marks)**

(a) Solve the following inequality, for $x \in \mathbb{R}, x \neq 1$:

$$\frac{3x+1}{x-1} \leq 6.$$

(b) **(i)** Write down a pair of real values of a and b, if $\log_3 4 = \dfrac{\log_2 a}{\log_2 b}$.

$a =$ $b =$

(ii) Show that $(\log_2 3)(\log_3 4) = 2$.

(iii) Hence, or otherwise, solve for $n \in \mathbb{N}$, the equation

$$(\log_2 3)(\log_3 4)(\log_4 5) \ldots (\log_{n-1} n)(\log_n (n+1)) = 11.$$

Question 3 **(30 marks)**

In this question, $i^2 = -1$.

(a) $z_1 = \dfrac{25(3-i)}{7-i}$. Write z_1 in the form $a + bi$, where $a, b \in \mathbb{R}$.

(b) $2 - i$ is a root of the equation $z^2 + az + b = 0$, where $a, b \in \mathbb{Z}$.
Find the value of a and the value of b.

(c) Use De Moivre's theorem to solve the equation $z^3 = 4 + 4\sqrt{3}i$.
Give each of your solutions in the form $r(\cos\theta + i\sin\theta)$, where $r \in \mathbb{N}$ and $0 \leq \theta \leq 2\pi$.

Question 4 **(30 marks)**

(a) Prove by induction that $13^n - 1$ is divisible by 12 for all $n \in \mathbb{N}$.

(b) $f(x) = x^3 - (k^2 + 1)x + k$, where $x, k \in \mathbb{R}$.

(i) Show that $f(k) = 0$.

(ii) Show that $x = 10$ is a root of the following equation, where $x \in \mathbb{R}$, **and** hence find the other two roots, in the form $p \pm \sqrt{q}$, where $p, q \in \mathbb{Z}$:

$$x^3 - 101x + 10 = 0.$$

Show:

Other roots:

Question 5 **(30 marks)**

(a) In this question part, p and $q \in \mathbb{Q}$.

(i) The following three terms are in arithmetic sequence:

$$p,\ q, \text{ and } \frac{1}{4}$$

Write q in terms of p.

(ii) The following three terms are in geometric sequence:

$$p,\ \frac{1}{4}, \text{ and } q$$

Write q in terms of p.

(iii) Using your results from **part (i)** and **part (ii)**, find the values of p.

(b) The general term of another sequence is $u_n = n[(n-2)!]$ for all $n \geq 2, n \in \mathbb{N}$.

Show that $u_{n+1} = (n-1)u_n + (n-1)!$

Question 6 **(30 marks)**

(a) The function g is defined as follows, for $x \in \mathbb{R}$:

$$g: x \mapsto 4x^3 - 40x^2 + 77x$$

Use calculus to find the co-ordinates of the local minimum point of g **and** the point of inflection of g.

Local minimum: (,) Point of inflection: (,)

(b) The function $f(a)$, where a and $x \in \mathbb{R}$, is defined as:

$$f(a) = \int_0^1 (x^2 - 2ax + a^2)\, dx.$$

(i) Find $f(5)$.

(ii) Find the minimum value of $f(a)$.

Section B	Contexts and Applications	150 marks

Answer **any three** questions from this section.

Question 7 **(50 marks)**

Caoimhe is repaying a loan of €20 000 over 5 years.
She makes equal monthly repayments of €389·56.
The first payment was made one month after the loan was issued.
The bank charges a fixed monthly interest rate of 0·526%.

(a) The table below shows some details on how the balance of the debt of €20 000 is reduced over the first four months. At the end of each month, interest of 0·526% on the previous month's balance is calculated – for example, the interest at the end of month 1 is €105·20.

This interest is paid out of Caoimhe's €389·56 monthly payment; the remainder of the €389·56 is then paid off the outstanding balance.

Complete the table.

Payment Number	Fixed Monthly Payment (€)	€389·56		Outstanding Balance (€)
		Interest at 0·526% (€)	Previous balance reduced by (€)	
0				20 000
1	389·56	105·20		
2	389·56		285·86	
3	389·56			19142·42
4	389·56			

(b) Find the amount that Caoimhe still owes at the end of the first two years of making repayments by finding the sum of the present values of the remaining 36 payments on the loan.

This question continues on the next page.

(c) Find, correct to 1 decimal place, the annual percentage rate (APR) that is equivalent to a monthly interest rate of 0·526%.

(d) After making monthly payments for three years, Caoimhe wants to finish repaying her 5-year loan in one more year. The bank advises her to take out a new 1-year loan to repay the outstanding €8761·77, making equal repayments at the end of each month, at an annual interest rate (APR) of 5·53%.

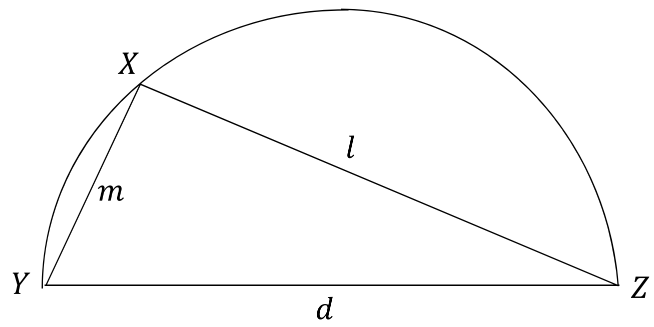
(i) Find the monthly interest rate that is equivalent to an APR of 5·53%.
Give your answer correct to 2 decimal places.

(ii) Find the amount that Caoimhe repays each month on the new 1-year loan.

(e) A sum of money, €A, is invested in a fund and is withdrawn after t years, where $t \in \mathbb{N}$.
The money is invested at a fixed APR of $6 \cdot 75\%$.
Find the least value of t needed so that the investment will amount to more than €$2A$ when it is withdrawn from the fund.

Question 8 **(50 marks)**

A triangle, XYZ is inscribed in a semi-circle of diameter d, as shown below.
$|XZ| = l$ and $|XY| = m$.
P is the length of the perimeter of the triangle.
A is the area of the triangle.



(a) **(i)** Find A, the area of triangle XYZ, in terms of l and m.

(ii) Express $l + m$ in terms of P and d.

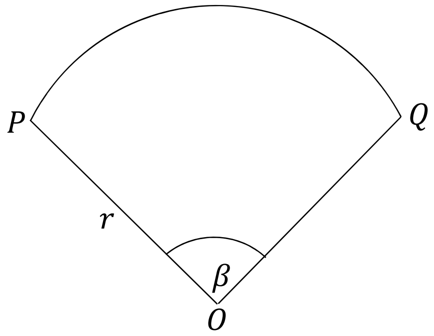
(iii) Using the identity $l^2 + m^2 = (l + m)^2 - 2lm$, or otherwise, show that:

$$d = \frac{P^2 - 4A}{2P}$$

(iv) Given that $A = 120$ cm^2 and $P = 60$ cm, use the result from **part (a)(iii)** to find the radius of the semi-circle.

This question continues on the next page.

(b) The diagram below (not drawn to scale) shows a sector POQ of a circle with centre O and radius r cm.
The angle POQ is β radians.
The area of the sector is 16 cm^2.
The perimeter of the sector is 20 cm.



(i) Show that r satisfies the equation:

$$r^2 - 10r + 16 = 0,$$

and find the roots of this equation.

Show:

Roots:

(ii) Find the corresponding values of β **and** explain why only one value is valid in the context of the question.

$\beta =$

Explanation:

Question 9 **(50 marks)**

(a) A lake was stocked with trout.
The number of trout in the lake after time t, can be approximated by the function:

$$N(t) = \frac{3000}{1 + 6{\cdot}5e^{-0{\cdot}814t}}$$

where t is the number of months since the lake was first stocked.
Use the function $N(t)$ to answer **parts (i)** to **(v)**.

(i) Find the number of trout in the lake when it was first stocked.

(ii) Find the approximate number of trout in the lake at the end of month 10.

(iii) Show that the number of trout in the lake has a limiting value of 3000.

(iv) Month n is the first month when the number of trout in the lake had risen above 2650. Find the value of n.

(v) Find in terms of t, the rate at which the number of trout in the lake is increasing.

This question continues on the next page.

(b) A different model to approximate the number of trout in the same lake uses the function

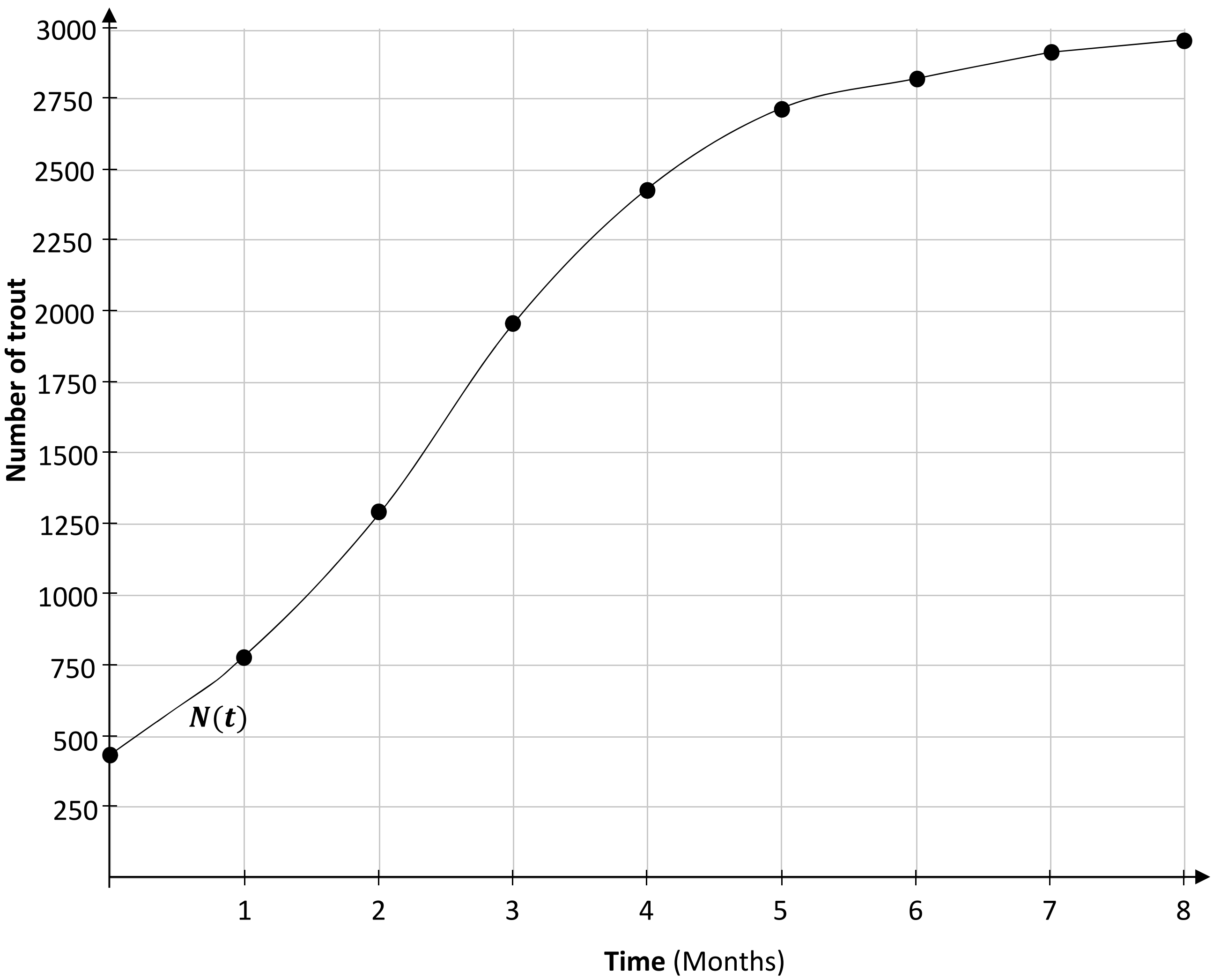
$$P(t) = 2600\left(\frac{15}{13} - e^{-0\cdot432t}\right)$$

where $t \in \mathbb{R}$ is the number of months since the lake was first stocked.

(i) Complete the table below (round each value to the nearest whole number).

t	0	1	2	3	4	5	6	7	8
$P(t)$	400					2700			

(ii) Draw the graph of $P(t)$ in the domain $0 \leq t \leq 8$, $t \in \mathbb{R}$ on the grid below.
Note: The graph of $N(t)$ is already plotted on the grid.



(iii) By referencing the graphs, describe one similarity and one difference between the two models.

Similarity:

Difference:

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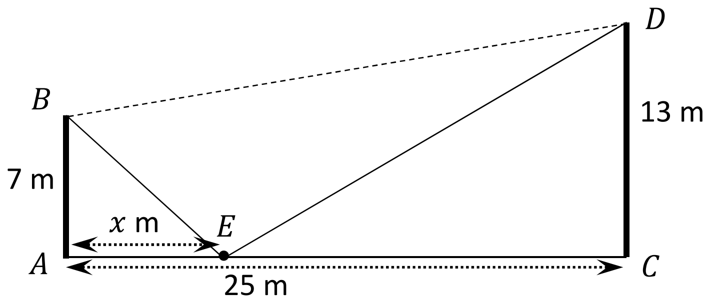
Question 10 **(50 marks)**

$[AB]$ and $[CD]$ are two vertical poles of height 7 m and 13 m, respectively.
They are 25 m apart on level ground, as shown on the diagram below.
The poles are held in place by a taut wire which is attached to the top of each pole (B and D) and also to the ground at a point E directly between the poles.
The point E is x m from the pole $[AB]$, where $x \in \mathbb{R}$.



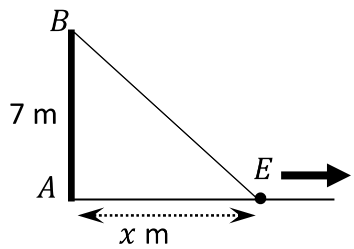
(a) **(i)** Find the distance $|BD|$, correct to 2 decimal places.

(ii) Show that the total length of the wire (from B to E to D) is:

$$\sqrt{x^2 + 49} + \sqrt{x^2 - 50x + 794}.$$

(iii) Find the two possible values of x for which $\angle BED$ is a right angle.
Give each answer correct to 2 decimal places.

(b) Part of the same diagram is shown on the right.
As the distance x increases, so does the distance $|BE|$.
x is increasing at a rate of 4 m per second.
Find the rate at which the distance $|BE|$ is increasing at the instant when the triangle ABE is isosceles.
Give your answer correct to 1 decimal place.



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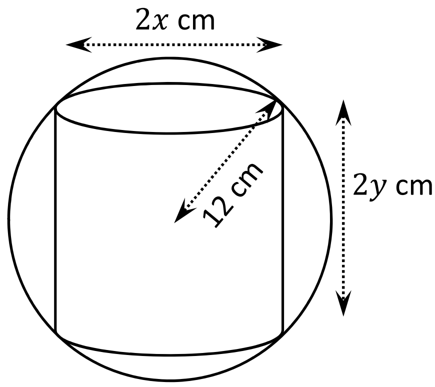
(c) The diagram on the right shows a right circular cylinder that just fits inside a sphere of radius 12 cm.
The diameter of the cylinder is $2x$ and its height is $2y$.



(i) Show that $x^2 = 144 - y^2$.

(ii) Write the volume of the cylinder in terms of y **and** hence find the volume of the largest right circular cylinder that fits inside a sphere of radius 12 cm.
Give your answer correct to the nearest cm^3.

Volume of cylinder in terms of y:

Volume of cylinder:

Page for extra work.
Label any extra work clearly with the question number and part.

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Leaving Certificate – Higher Level

Mathematics - Paper 1

2 hours 30 minutes